Tina's Logarithmic Spiral

The Logarithmic Spiral is the "Spira Mirabilis" beloved of Jacob Bernoulli a famous seventeenth century mathematician.

This spiral has many marvellous properties but the one which concerns me is its use as a slide rule calculator. Two such devices, the Keuffel and Esser Log Spiral and the Tyler Slide Rule were described by Rodger Shepherd in the Journal of the Oughtred Society, Volume 9.1. The Nystrom Calculator which is also, I believe (writer's opinion), based on the Logarithmic Spiral is described in the Journal of the Oughtred Society, Volume 4.2 and 16.2.

Those of you who have read my previous articles concerning home built Thacher, Bygrave and Complex Number slide rules will not be surprised that when faced with the fact that I am unlikely to obtain any of the above devices any time soon I chose to make one so that I could explore the devices functionality.

A bit of mathematics

The basic equation of a Logarithmic Spiral is $R = ae^{\frac{\theta}{b}}$ .... Equation 1.

Where R is the length of a radial line, the distance from a point on the spiral to the origin

e is approximately equal to 2.71828 and is the basis of natural logarithms

a and b are constants which effect the shape of the spiral

Θ is the angle in radians between the baseline and the radial line

From the above equation it can be seen that if we multiply two values R1 and R2 represented by radial lines of length R1 and R2 and angles from the base line of Θ1 and Θ2 then

$$R1 * R2 = a^2 * e^{\frac{\theta 1}{b}} * e^{\frac{\theta 2}{b}} = a^2 * e^{\frac{\theta 1 + \theta 2}{b}}$$

This gives :- $b * ln((R1 * R2)/a^2)$ = (Θ1+ Θ2)

Or b*(ln(R1)+ln(R2)-2*ln(a)) = (Θ1 + Θ2) ....Equation 2

Therefore the two angles added together are proportional to the natural logarithms of the two values added together. This means that adding the angles of the radial lines on a logarithmic spiral can achieve the same result as adding the logarithms on a standard slide rule.

Initial Design

I don't like the constant 'a' in the above equations it makes them unnecessarily messy therefore for my design I am going to say that a = 1.

This gives the equation b*ln(R1*R2)= Θ1+ Θ2 as the basis of my design.

Now I want my spiral to be sufficiently large for ease of use but not too large and I want to start the spiral at R1 = 1 unit at angle ϴ1=0 and finish the spiral at R2 = 10 units at angle ϴ2 = 2π Radians (360 Degrees).

Using the initial Equation 1:- $R=ae^{\frac{\theta}{b}}$ with a = 1 and with R=10 and ϴ = 2π I can find the missing constant b which I need for my design.

This gives $10=e^{\frac{2\pi}{b}}$ .....Equation 3

Therefore b = 2π/ln(10) = 2.7288 to 5 significant figures

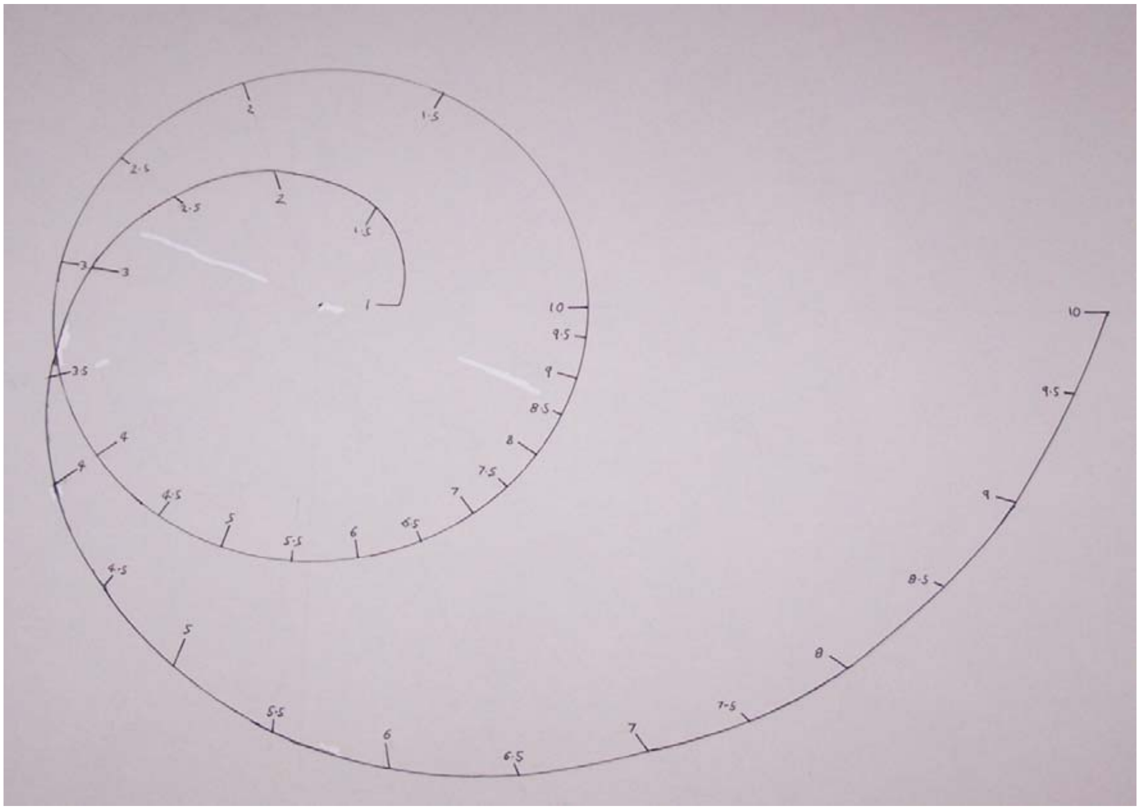
So the equation of my spiral is $R = e^{\frac{\theta}{2.7288}}$

So now I can make a table of values of R for given angles ϴ to help me construct the spiral.

Angle (Radians)	Radial Length (Units)
0	1
π/8 (22.5°)	1.1548
π/4 (45°)	1.3335
3π/8 (67.5°)	1.5399
π/2 (90°)	1.7783
5π/8	2.0535
3 π/4 (135°)	2.3714
7π/8	2.7384
π(180°)	3.1623
9π/8	3.6517
5π/4 (225°)	4.2170
11π/8	4.8697
3π/2 (270°)	5.6234
13π/8	6.4938
7π/4 (315°)	7.4989
15π/8	8.6596
2π (360°)	10

I am arbitrarily going to set one unit to equal 3cm so that my spiral fits onto an A3 sheet.

The drawing of the scale is shown below. I am not claiming that the drawing is accurate; it has been made to demonstrate the principles not as a marketable device. The spiral has been plotted and the positions of the radial lines with integer and half integer lengths marked on the spiral. The points at which these radial lines intersect the circumference of a circle are also marked on that circle. This demonstrates an alternative method of creating a circular slide rule scale.



Now I am sure that some of you have noticed that the radial lengths in the above table exactly match the numbers found on the C scale of any standard logarithmic slide rule. If you have a slide rule with a log scale on the front (a Thornton AD070 for example) you will be able to read off the values in the above table directly. At the point 0.25 on the log scale you will find the value 1.7783, the length of the radial line at $\pi/2$ radians. At the point 0.5 on the log scale you will find the value of 3.1623 (10^0.5), the length of the radial line at π radians. The C scale is a scale of radial line lengths for my Logarithmic Spiral. Since there is a direct one to one relationship between the C scale on any slide rule and the radii of this logarithmic spiral the fact that a logarithmic spiral can be used as a calculator comes as no surprise.

Returning to Equation 1 it soon becomes clear what is happening.

Again with a = 1, $R=e^{(\Theta/b)}$ where $b = 2\pi/\ln(10)$

This gives $\ln(R) = \Theta*\ln(10)/2\pi$

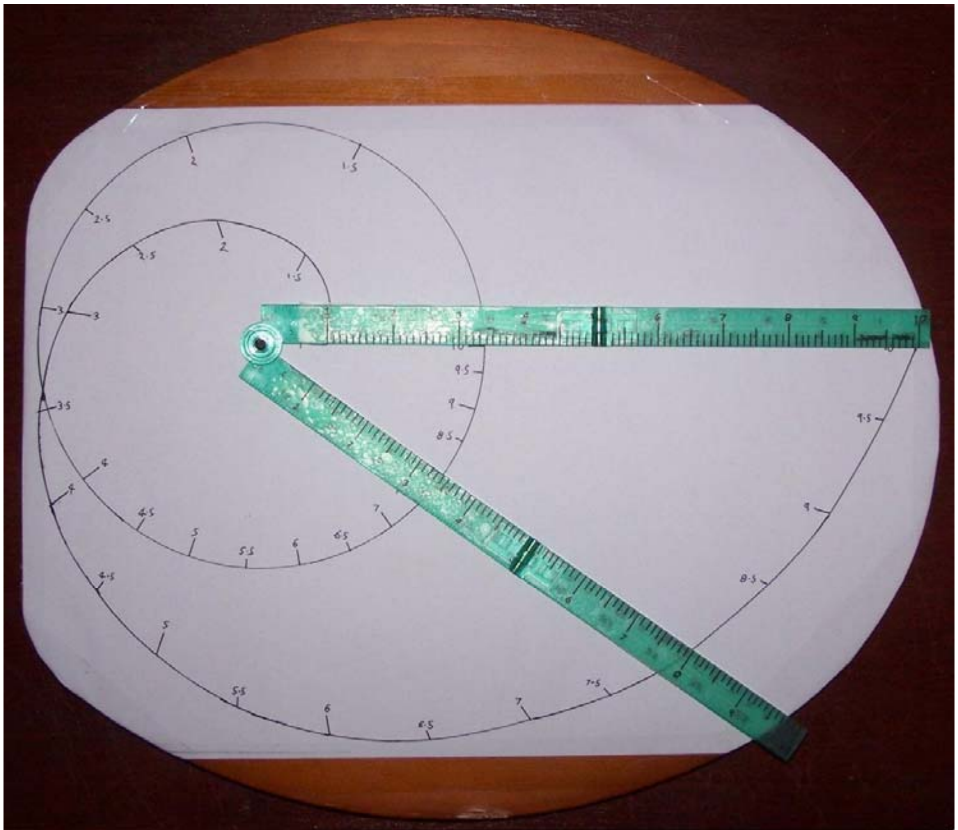
And $\ln(R)/\ln(10) = \Theta/2\pi$ which since $\ln(R)/\ln(10)= \log(R)$ gives $\log(R) = \Theta/2\pi = \Theta/360$ (if you are happier working in degrees and you provide Θ in degrees)

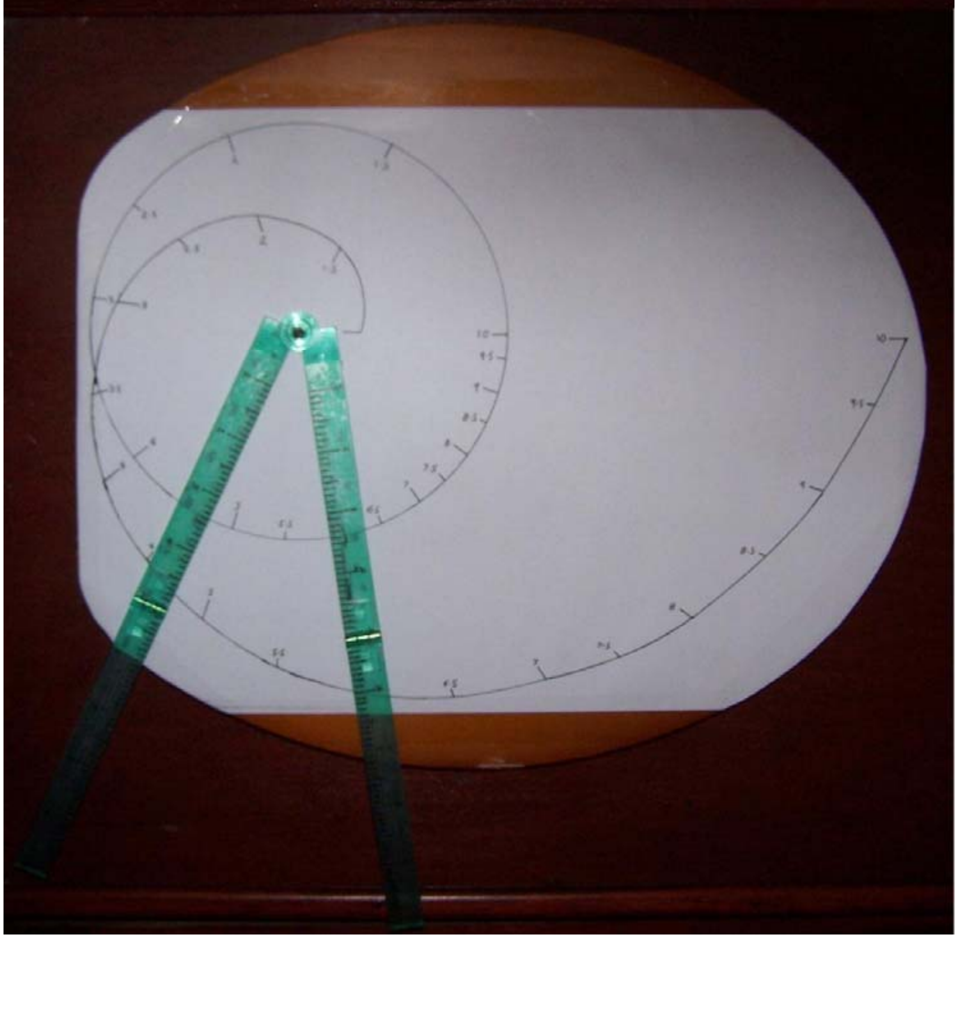
Where $\Theta/2\pi$ is the fraction of the circle between the baseline and the radial line.

So for my logarithmic spiral the logarithm to base 10 of the radial length equals the fraction of the circle between the baseline and the radial line.

On a normal linear slide rule the logarithm to base 10 of a number on the C scale equals the distance from the number 1 to that number divided by the length of the scale.

The similarity can be seen.





In the above image two cursors with a common axis are used. The picture shows the setting for multiplying 8 by 6. Ignore the numbers on the spiral, all multipliers are set in and answers read on the linear scales on the cursors. In the top image one cursor is lined up with the baseline, the other is lined up so that 8 on the cursor coincides with the log spiral curve. The two cursors are then rotated without changing the angle between them until the number 6 on the top cursor coincides with the spiral curve. The answer, 48 is read on the linear scale of the other cursor where it crosses the spiral curve. Yes it has been constructed on an old toilet seat, one uses what one has available in these experiments.

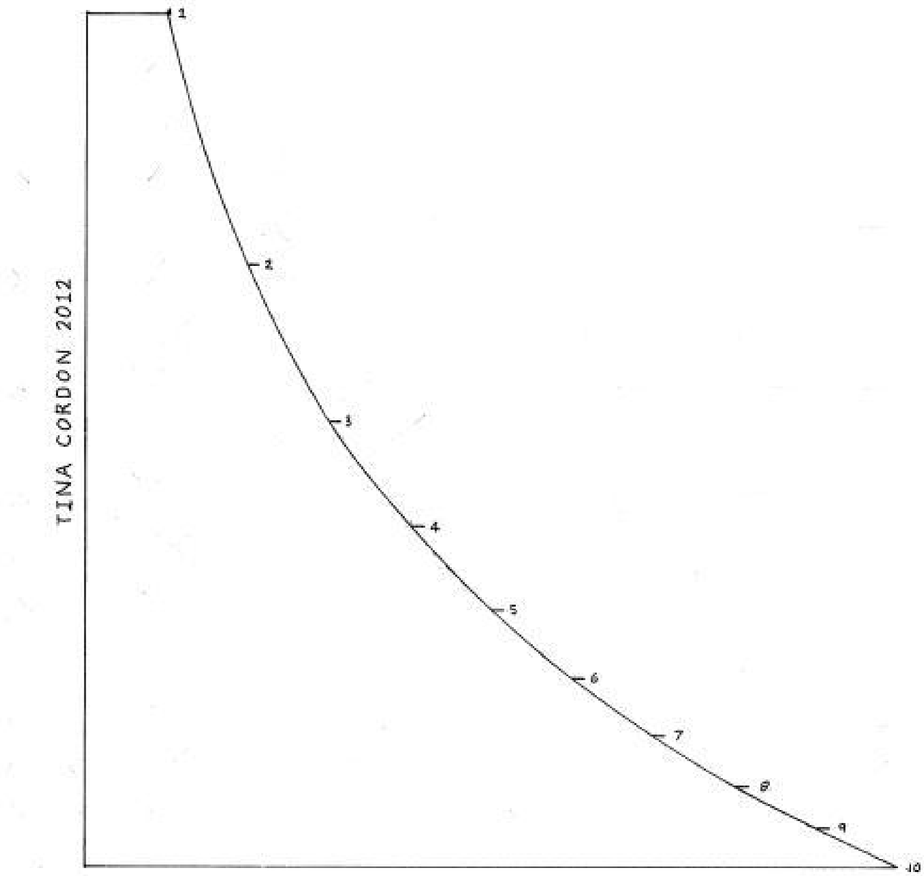
<u>Benefits of using a Logarithmic Spiral for calculation</u>

As shown previously calculating with a logarithmic spiral is less efficient than calculating with a circular slide rule, however, just as the use of a linear slide rule gives one a feel for the mathematical principles on which it is based, the same is true for the Logarithmic Spiral. One of the advantages of the logarithmic spiral is that it gives results on a linear scale not a logarithmic scale therefore linear interpolation between scale lines is inherently more accurate. Using linear scales also provides the opportunity of using vernier scales to obtain an extra significant figure of accuracy.

<u>Cylindrical Logarithmic Spiral</u>

Thinking laterally as usual I decided to find out what would happen if I arbitrarily changed equation 2 so that the angles were replaced by distances from a baseline.

The resulting curve is shown below.

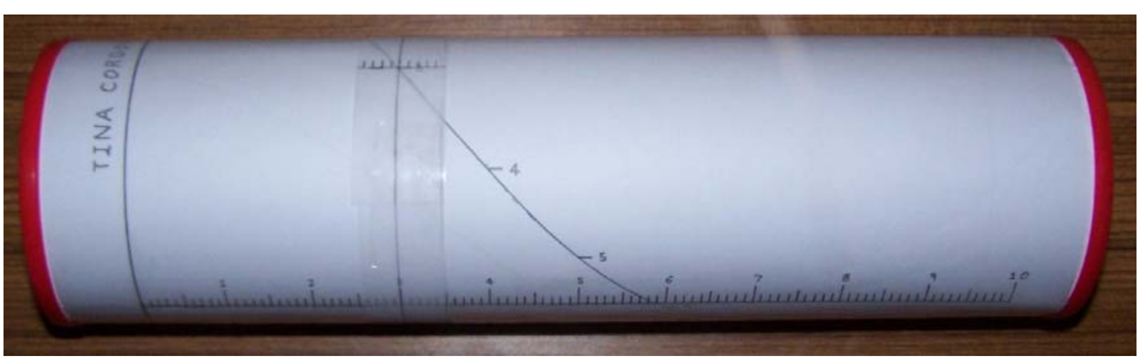


Mathematically this curve is very simple. It is just a plot of log(x) on the vertical scale against x on the horizontal scale.

When this curve is placed on the surface of a cylinder, all that is then needed are appropriate cursors.

The main cursor should be celluloid and span the cylinder entirely so that it cannot move sideways with respect to the curve and should have a linear scale drawn on its surface.

The minor cursor is a narrow strip of celluloid with a line drawn down its centre and a crosshair. The result is shown below.



To Multiply

Multiplication with this slide rule is very simple. Move the main cursor until the first multiplier on its linear scale coincides with the curve. The minor cursor is then moved so that it is on the baseline through the number 1 and its centre line passes through the second multiplier. Rotate the main cursor until the minor cursor crosses the curve. The answer is found on the linear scale at the point at which it crosses the curve.

As with a standard linear slide rule this rule can be used in reverse when results are greater than 10.

To Divide

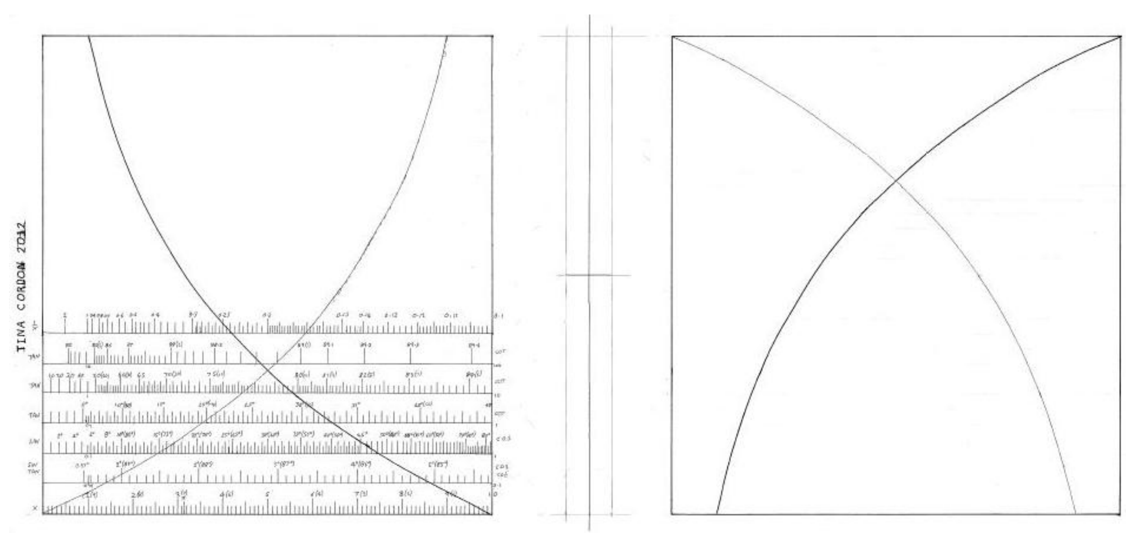
Place the main cursor so that the dividend on the linear scale coincides with the curve as shown above (6). Place the minor cursor on the curve so that its centre line passes through the divisor (3 in the picture. Rotate the major cursor to bring the minor cursor back to the baseline and the result is found where the linear scale on the major cursor crosses the curve.

Second Prototype

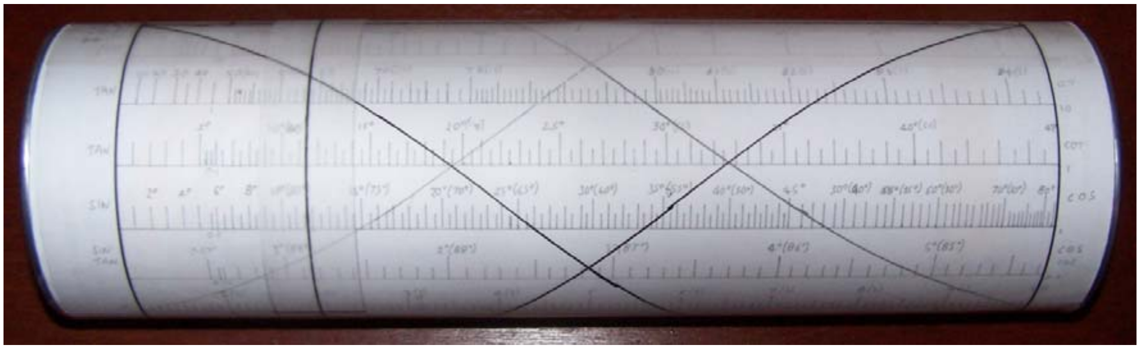
By placing the linear scale on the stock instead of the cursor construction becomes simpler and use slightly more complicated. By adding other scales, tied to the linear scale other calculations can be done simply. The scales I have added are Sine (Cosine) scales, Tangent (Cotangent) scales and a Reciprocal scale. The only limitation on the scales added is the amount of space on the stock. I have not added scales of Squares or Square Roots because these values are easily obtained with a slight modification of the Main Cursor.

Tyler's Slide Rule utilised a second Spiral Curve which was a mirror image of the first to obtain Squares and Square Roots directly. This same facility is available using my curve by placing an inverted curve on the main cursor.

By adding curves reflected on the vertical plane as well and adding complimentary numbers to the linear scale it also becomes simple to find the results $1 - X^2$ and $(1 - X^2)^{0.5}$.



The drawing to the left in the above image shows the design for the base scale. The drawing to the right shows the design for the major cursor.

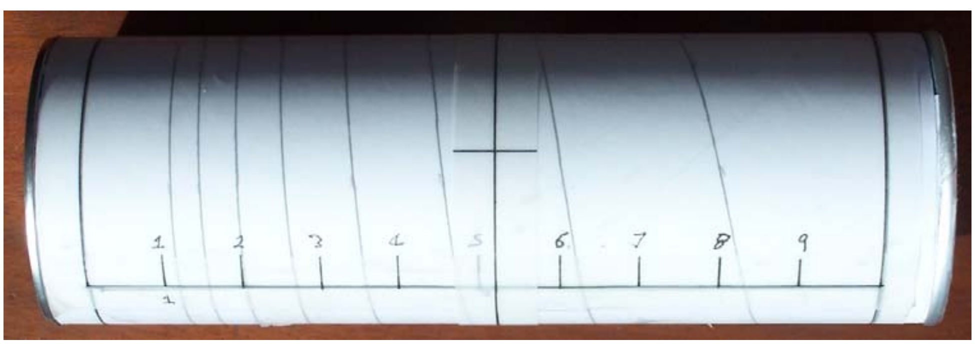


This second design works very well. The accuracy is limited by the accuracy of my drawing and the length of the log curve. By adding a vernier scale to the minor cursor I can however get an extra significant figure when reading the answer.

<u>Third Prototype</u>

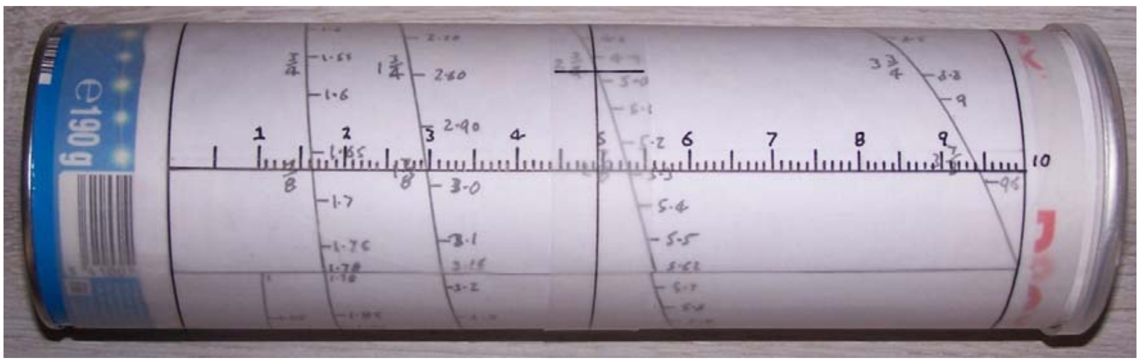
Having read Conrad Schure's description of his Nystrom Calculator in the Journal of the Oughtred Society and Nystrom's own description in his book, "A Treatise on Screw Propellers and their Steam Engines" I am convinced that John W. Nystrom took the next step. Looking at the pictures available I believe that Nystrom used a very long Logarithmic Spiral chopped into sequential segments confined between concentric circles on the surface of his Calculator.

I, therefore have done a similar thing with my cylindrical rule. I have stretched the log curve so that it wraps around the cylinder eight times. I chose eight for simplicity of construction. The resulting cylindrical slide rule is shown below.



Multiplication and division are achieved in the same way as with the first prototype. Theoretically the slide rule should be eight times more accurate as a result of the increased curve length but there are inherent problems with this design. As before the answer is read on the baseline where the curve crosses that line but now it crosses that line eight times giving eight possible answers. Other long scale slide rules both linear and circular exhibit the same problem. In order to use it effectively it is necessary to have a good idea of the approximate result otherwise the wrong answer will be chosen. It is possible using knowledge of how many turns around the cylinder are the multiplier and multiplicand to find the correct answer without approximation. For example the number 2 is found two turns of the scale and a fraction from the origin; the number 3 is found 3 turns and a different fraction from the origin therefore the answer will be on either the fifth turn or sixth turn depending on the size of the fractions. Personally I find this far too cumbersome and I found a way of automating this for the fourth prototype. The second problem is that in stretching the curve it has become almost parallel with the centreline of the minor cursor making it difficult to accurately set the cursors.

Fourth Prototype



In order to address the problems stated I have added the scale to the logarithmic line so that errors in setting the rule are decreased. The number and fraction to the right of each section of the curve is the section number and approximate fraction of each section on which the multipliers are found. On this prototype I have wound the Log Line around the cylinder four times, by adding the location numbers I can use a bit of clock arithmetic to find the approximate position of the answer. This prototype has been created using a Pringles tube as you might have guessed. I have used the cap of the Pringles tube to make the clock scale shown below. By adding the approximate positions of the multipliers on the clock dial the approximate position of the answer is found.

For example 6 is on the third log section about 1/8th of a rotation along that section. 7 is also on the third section about 3/8th of a rotation along. Using the dial 3 + 3 = 2 according to the rules of clock arithmetic and 1/8th + 3/8th = ½, therefore the answer will be found about half way along the section labelled 2 where 4.2 is indeed found.

This is a much more practical calculator.